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THE BASIS FOR AN ELASTIC-PLASTIC CODE.(U)

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THE BASIS FOR AN ELASTIC-PLASTIC CODE*

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A complete stress-analysis of a metal-forming process is necessary in order to assess the onset of metal forming defects such as the initiation of ternal or surface cracks or the generation of residual stresses, and this emands elastic-plastic analysis. This report comprises comments on the plasticity theory formulation needed for a finite-element computer code designed or the analysis of metal forming processes. J. R. Rice has pointed out the portance of convection and rotation terms in the definition of stress-rate or inclusion in elastic-plastic constitutive relations, and that the development of uniqueness and stability analysis by R. Hill provides a convenient vehicle for including these effects. The development of these concepts is described and how they generate a convenient variational principle to form the asis of an elastic-plastic code to analyze metal forming.

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The Basis for an Elastic-Plastic Code

1. Introduction

For satisfactory analytical assessment of metal forming problems, it is necessary to evaluate the varying stress distribution throughout the work-piece since forming defects, such as the development of internal cracks, depend on stress history, and residual stresses can be important in deciding the utilization of a formed part. Elastic characteristics usually play an essential role in the determination of stress, even in combination with extensive plastic flow which may involve strains a thousand times elastic strain magnitudes. Thus analysis of metal forming problems for such assessments must be based on elastic-plastic theory. The same is true for other stress analysis problems when plastic flow occurs, unless plastic flow is occurring simultaneously throughout the entire body throughout the duration of the process, in which case plastic-rigid analysis is adequate.

Because plasticity laws are incremental in nature, they result in relations between stress-rate and strain-rate, or equivalently in numerical evaluations, between stress and strain increments. For the rate-independent laws usually adequate at temperatures low compared with the melting temperature, linear relations between stress-rate and strain-rate arise. When plastic flow is taking place, the coefficients are functions of the current stress for the common laws and below the yield

stress the elastic laws apply in incremental form. Because of the structure of these laws, elastic-plastic problems are commonly solved in terms of equations for stress-rates and strain-rates, containing stresses as coefficients. Consider the solution to have been computed up to the time t. After a time step forward, Δt, the solution for rates gives the stress at time $t + \Delta t$ in the form $\underset{\sim}{\sigma}(t) + \dot{\underset{\sim}{\sigma}}\Delta t$, and similarly for other variables. $\dot{\underset{\sim}{\sigma}}$ is the appropriate stress-rate. Then a new time step can be taken and the process repeated.

Extensive studies of the application of such laws to stability and uniqueness of solutions have been made by Hill (see, for example [1], [2], [3]* where he shows that care in the selection of stress definitions and stress-rate and strain-rate expressions is important for a satisfactory development of the theory. Rice [4] has pointed out that such questions are also important in developing a satisfactory theoretical basis for elastic-plastic stress analysis, particularly in the common circumstance that the tangent modulus in plastic flow is of the order of the stress. Convected and rotation terms then become important in the stress-rate expression, and analogously stress variables should be selected so that the influence of rate of deformation of the boundaries of the body does not complicate the variational principle commonly used to replace the equilibrium equations for evaluation of solutions. This requirement can be achieved by using the unsymmetric nominal stress

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*Numbers in square brackets refer to the bibliography.

(Piola-Kirchoff I) in which the stress is defined as force per unit undeformed area. The variational principle then involves an integral over the undeformed body which is fixed.

Plastic flow is essentially a fluid type phenomenon which can be most conveniently expressed in terms of the current configuration of the material. Thus a reference configuration which remains invariant throughout the motion is not appropriate and so the configuration at time t is adopted as the reference state for evaluation of the deformation from t to $t + \Delta t$, where Δt is sufficiently small for adequacy of first order theory.

The framework described above provides a satisfactory foundation for a finite-element elastic-plastic code as discussed by McMeeking and Rice [5]. In effect, by choosing the current configuration as the reference state the Cauchy stress (or true stress in Cartesian coordinates) the unsymmetric nominal stress (Lagrange or Piola-Kirchhoff I) and the symmetric nominal stress (Kirchhoff or Piola-Kirchoff II) all have identical values at the current time which simplifies utilization of the appropriate stress for the appropriate component of the calculation. Although the stress components themselves are identical, rates of change of the different stresses are not the same.

2. Development of the Theory

Following Hill [2] and using for the most part his notation, we consider the unsymmetric nominal stress (variously referred to as Lagrange or Piola-Kirchoff I) s_{ij} defined so that the jth component of the force transmitted across a deformed element, which in the initial or

reference state had area $d\overset{\circ}{S}$ and unit normal $\overset{\circ}{\nu}_i$, is

$$d\overset{\circ}{S}\,\overset{\circ}{\nu}_i\, s_{ij} = dF_j \tag{1}$$

Hill considers (p. 214 of [2]) rate or flow type constitutive laws of the form

$$\dot{s}_{ij} = \frac{\partial E}{\partial(\partial v_j/\partial X_i)} \tag{2}$$

where X_i are rectangular Cartesian coordinates in the initial or reference configuration, E is a homogeneous function of degree two in the velocity gradients, $\partial v_j/\partial X_i$, and where $\dot{s}_{ij}$ is the partial time derivative at fixed $\underset{\sim}{X}$, i.e. a material derivative at a particle. The velocity $v_j(\underset{\sim}{X},t)$ gives the distribution at time t expressed in the initial coordinates of the corresponding material points (note that a tilde under a symbol denotes a vector or tensor in absolute notation).

Boundary value problems are considered in which for a volume $\overset{\circ}{V}$ in the reference state, at time t stress rates $\dot{s}_{ij}$ $(\underset{\sim}{X},t)$ and velocities $v_i(\underset{\sim}{X},t)$ are sought for prescribed nominal traction rates $\dot{F}_j$ over the part of the surface $\overset{\circ}{S}_F$, velocity v_j over the remainder of the surface $\overset{\circ}{S}_V$ and body force rates $\dot{g}_j$ per unit initial volume. Then the variational principle

$$\delta\Big[\int_{\overset{\circ}{V}} E\, d\overset{\circ}{V} - \int_{\overset{\circ}{S}_F} \dot{F}_j v_j\, d\overset{\circ}{S} - \int_{\overset{\circ}{V}} \dot{g}_j v_j\, d\overset{\circ}{V}\Big] = 0 \tag{3}$$

in the class of continuous differentiable velocity fields satisfying the velocity boundary condition on $\overset{\circ}{S}_V$, characterizes the solution, for it yields the equilibrium equations

$$\frac{\partial \dot{s}_{ij}}{\partial X_i} + \dot{g}_j = 0 \tag{4}$$

and the boundary traction rate condition

$$\overset{\circ}{\nu}_i \dot{s}_{ij} = \dot{F}_j \tag{5}$$

for nominal stress, s_{ij} , and the reference geometry.

In writing the elastic-plastic constitutive relation, we wish to associate the velocity gradient in (2) with the rate of deformation or velocity strain:

$$D_{ij} = \tfrac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{6}$$

where x_i are Cartesian coordinates expressing the position of particles in the deformed body

$$x_i = x_i(\underset{\sim}{X};t) \tag{7}$$

Thus to permit simultaneous use of (3) and the plasticity laws expressed in the usual form of rate of deformation of the current configuration, Hill takes the current configuration to be the reference state, and hence

$$X_i = x_i(\underset{\sim}{X}, t) \tag{8}$$

for a particular t . The theory is expressed in this form for evaluation of $\dot{s}_{ij}$ and v_j and hence the solution and configuration at $t + \Delta t$, which provides a new reference state for evaluation of the next time step Δt .

Note that at the instant t, when the current and reference configurations are identical, the nominal stress components s_{ij} are equal to the Cauchy or true stress components σ_{ij}, so that at this instant s_{ij} is symmetric.

The device of selecting the configuration at time t to be the reference state for evaluation of the solution at time $t + \Delta t$ thus permits simultaneous use of the convenient variational principle (3) in terms of nominal stress s_{ij} and a fixed geometry and the familiar plasticity laws expressed in terms of the Cauchy, or true stress, σ_{ij}.

By working with curvilinear convected coordinates, ξ^{α}, having an arbitrary configuration in the reference or initial state, Hill ([2], p. 219 ff.) shows that the rate potential (2) follows from associated rate potentials for other stresses and stress rate expressions, some of which are more convenient for representing elastic-plastic laws.

Consider curvilinear coordinates ξ^{α} in the initial or reference state with base vectors $\overset{\circ}{\underset{\sim}{g}}_{\alpha}$. After deformation as shown in Fig. 1 these become $\xi^{\alpha'}$, having the same values for the same material particles as ξ^{α}. Then $\xi^{\alpha'}$ are the convected coordinates and the corresponding base vectors are $\underset{\sim}{g}_{\alpha'}$, which are $\overset{\circ}{\underset{\sim}{g}}_{\alpha}$ deformed by the motion. Then the Cauchy stress tensor $\underset{\sim}{\sigma}$ has contravariant components $\sigma^{\alpha'\beta'}$ in convected coordinates such that the force $d\underset{\sim}{F}$ transmitted across an element of the deformed body of area dS and unit normal vector $\nu_{\alpha'}$ is given by:

$$d\underset{\sim}{F} = \sigma^{\alpha'\beta'} (\nu_{\alpha'} dS) \underset{\sim}{g}_{\beta'} \tag{9}$$

Since the primed coordinates are made evident by the notation ν and $\underset{\sim}{g}$ for normal and base vectors in the deformed state, the primes will usually be dropped hereafter, for $\xi^{\alpha} = \xi^{\alpha'}$ denote the same material point.

For the Lagrange or Piola-Kirchhoff I stress, s_{ij}, the force computed in the reference frame is that actually acting across the deformed element (see Fung [6] p. 437 for the usual definition in Cartesian coordinates). For the present consideration of convected coordinates, the expression for $d\underset{\sim}{F}$ thus takes the form

$$d\underset{\sim}{F} = s^{\alpha\beta}(\mathring{\nu}_{\alpha}\, d\mathring{S})\mathring{\underset{\sim}{g}}_{\beta} \tag{10}$$

For the Kirchhoff or Piola-Kirchhoff II stress, τ_{ij}, Fung points out that the force vector computed in the reference frame must be transformed by the motion to give the actual force across the deformed element, so that for convected coordinates

$$d\mathring{\underset{\sim}{F}} = \tau^{\alpha\beta}(\mathring{\nu}_{\alpha} d\mathring{S})\mathring{\underset{\sim}{g}}_{\beta}$$

becomes

$$d\underset{\sim}{F} = \tau^{\alpha\beta}(\mathring{\nu}_{\alpha} d\mathring{S})\underset{\sim}{g}_{\beta} \tag{11}$$

Nanson's relation for area element:

$$\rho\nu_{\alpha} dS = \mathring{\rho}\mathring{\nu}_{\alpha} d\mathring{S} \tag{12}$$

then gives, using (9) and (11)

$$\tau^{\alpha\beta} = \frac{\mathring{\rho}}{\rho}\,\sigma^{\alpha\beta} = J\,\sigma^{\alpha\beta} \tag{13}$$

where J is the Jacobian of the transformation from the reference to the deformed state, i.e.

$$\underset{\sim}{g}_\alpha \cdot \underset{\sim}{g}_\beta \times \underset{\sim}{g}_\gamma = J(\overset{o}{\underset{\sim}{g}}_\alpha \cdot \overset{o}{\underset{\sim}{g}}_\beta \times \overset{o}{\underset{\sim}{g}}_\gamma) \tag{14}$$

Fig. 2 shows the particular situation when the ξ^α coordinates in the reference frame are Cartesian, X^α. The Cartesian coordinates representing points in the deformed body according to the point transformation, equ. (7), are x^i. The usual definition of the Kirchhoff or Piola-Kirchhoff II stress (see Fung [6] p. 439) is given in terms of this point transformation. From equ. (7), define the deformation gradient

$$\underset{\sim}{F} = \frac{\partial x^i}{\partial X^\alpha} \tag{15}$$

Then the Piola-Kirchhoff II stress $\underset{\sim}{\tau}$ is given in terms of the Cauchy stress in Cartesian coordinates $\underset{\sim}{x}$, $\overset{c}{\sigma}$ by

$$\underset{\sim}{\tau} = J\, \underset{\sim}{F}^{-1}\, \overset{c}{\underset{\sim}{\sigma}}\, \underset{\sim}{F}^{-1^T} \quad ([7]\text{ p. } 125) \tag{16}$$

where $J = \det(\underset{\sim}{F})$. This is in accord with (13) where $\sigma^{\alpha\beta}$ are the contravariant components of the Cauchy stress with respect to the convected coordinates $X^{\alpha'}$, for if $\overset{c}{\sigma}{}^{ij}$ are the Cartesian coordinates of $\underset{\sim}{\sigma}$ with respect to $\underset{\sim}{x}$, then the tensor change of variables law gives

$$\sigma^{\alpha'\beta'} = \frac{\partial X^{\alpha'}}{\partial x^i} \frac{\partial X^{\beta'}}{\partial x^j} \sigma^{cij} \tag{17}$$

in terms of the coordinate transformation from $\underset{\sim}{x} \rightarrow \underset{\sim}{X}'$ in the deformed geometry. Because of the property of convected coordinates that $X^\alpha = X^{\alpha'}$

for the same particle, (7) also expresses the coordinate transformation, and (13) and (17) are seen to be equivalent to (16). This connection has been pointed out by Nemat-Nasser [8]. Note that $s^{\alpha\beta}$ and $\tau^{\alpha\beta}$ are tensor densities and not absolute tensors, so that equations such as (2) are not pure tensor relations.

For rate independent constitutive relations the rate-potential function E in (2) is a homogeneous function of degree two. For the choice that the reference state is the current state, (2) follows from the existence of an associated rate potential function $F(\varepsilon_{\alpha\beta})$, of the strain-rate components

$$\varepsilon_{\alpha\beta} = \tfrac{1}{2}\,(v_{\alpha,\beta} + v_{\beta,\alpha}) \tag{18}$$

where $\underset{\sim}{v}$ is the velocity vector in the deforming body and the comma denotes covariant differentiation. This function generates $\dot{\tau}^{\alpha\beta}$ through

$$\dot{\tau}^{\alpha\beta} = \frac{\partial F}{\partial \varepsilon_{\alpha\beta}} \tag{19}$$

where the superposed dot indicates time derivative of the convected components or convected derivative. This is equivalent to the partial time derivative at fixed X, or a material type derivative. The structure of (19) indicates that since $\underset{\sim}{\tau}$ is a tensor density, F is a scalar density and not an absolute scalar invariant.

A derivation of a relation needed in the following analyses to establish (2) is given in the Appendix. It is the relationship between $\dot{s}^{\alpha\beta}$ and $\dot{\tau}^{\alpha\beta}$, which, for the particular choice of reference state mentioned, takes the form

$$\dot{s}^{\alpha\beta} = \dot{\tau}^{\alpha\beta} + \tau^{\alpha\gamma} v^{\beta}_{,\gamma} \tag{20}$$

Now from (19) and (20)

$$\dot{s}^{\alpha\beta} = \partial F/\partial(\epsilon_{\alpha\beta}) + \tau^{\alpha\gamma} v^{\beta}_{,\gamma}$$

multiplying both sides by $v_{\beta,\alpha}$ gives

$$\dot{s}^{\alpha\beta}\, v_{\beta,\alpha} = [\partial F/\partial(\epsilon_{\alpha\beta})]\epsilon_{\alpha\beta} + \tau^{\alpha\gamma} v^{\beta}_{,\gamma}\, v_{\beta,\alpha} \tag{21}$$

using (18) and the fact that $\tau^{\alpha\beta}$ is symmetric. Now $F(\epsilon_{\alpha\beta})$ is a homogeneous function of degree two since for plasticity (19) is rate independent, hence Euler's theorem for homogeneous functions permits us to write (21) in the form

$$\dot{s}^{\alpha\beta} v_{\beta,\alpha} = 2F(\epsilon_{\alpha\beta}) + \tau^{\alpha\gamma} v^{\beta}_{,\gamma}\, v_{\beta,\alpha} \tag{22}$$

Again using Euler's theorem, this is consistent with (2) in terms of convected coordinates if

$$\dot{s}^{\alpha\beta} = \partial E/\partial\ (v_{\beta,\alpha}) \tag{2a}$$

and

$$2E(v_{\beta,\alpha}) = 2F(\epsilon_{\alpha\beta}) + \tau^{\alpha\gamma}\ v^{\beta}_{,\gamma} v_{\beta,\alpha} \tag{23}$$

Equation (21) is a contracted scalar relation based on the tensor expression (20) and hence does not validate the tensor relation (2a) or (2). It merely prescribes the form of the rate-potential function E if such a function exists. Substitution of (23) into (2a) does yield (20) and hence establishes E as a homogeneous, second

degree, rate-potential function of $v_{\beta,\alpha}$ for the nominal stress rate $\dot{s}^{\alpha\beta}$.

A similar argument for the Cauchy stress $\sigma^{\alpha\beta}$ yields a trial potential function, which however fails to yield the correct expression for $\dot{\sigma}^{\alpha\beta}$ from the rate-potential equation. Hence a rate potential function does not exist in this case.

The laws of plasticity are normally obtained by measuring "true stress" and increments of strain defined in terms of Cartesian coordinates in the current configuration without rotations occurring. Since the theory must apply in the presence of rotations, their influence must not affect the stress rate term in the constitutive relation, hence a spin-invariant stress rate is needed, such as the Jaumann rate. We will work in terms of the configuration at time t , which is the reference configuration, and utilize Cartesian coordinates $\underset{\sim}{x}$.

In formulating the finite element theory for numerical analysis of elastic-plastic problems, we wish to use the variational relation (3) in terms of $\dot{s}_{ij}$ and a constitutive relation associated with (19) in terms of τ_{ij} since it can be conveniently associated with measurements of plasticity laws. We have seen that a law in the form (19) implies the validity of (2) and hence the variational principle. As pointed out in [5] this structure in terms of τ_{ij} leads to symmetric stiffness matrices in the finite element formulation which simplifies the numerical procedures.

Now the relationship between the nominal stress $\underset{\sim}{s}$ and the Cartesian true stress $\underset{\sim}{\overset{c}{\sigma}}$ is

$$\underset{\sim}{F}\,\underset{\sim}{s} = J\,\underset{\sim}{\overset{c}{\sigma}} \tag{24}$$

see, for example [7], p. 125, where the nominal stress is defined as the transpose of $\underset{\sim}{s}$ (or it can be deduced from (9), (10) and (17)). Taking the material derivative of (24), and noting that F and J are unity for coincident reference and current configurations, one obtains

$$\dot{s}_{ij} + s_{kj}\frac{\partial v_i}{\partial x_k} = \overset{c}{\sigma}_{ij}\frac{\partial v_k}{\partial x_k} + \dot{\overset{c}{\sigma}}_{ij} \tag{25}$$

The difference between the Jaumann derivative of $\underset{\sim}{\overset{c}{\sigma}}$ and its material derivative is the contribution of the rate of rotation of the axes which rotate with the body according to the anti-symmetric tensor angular velocity expression

$$\left.\frac{\partial v_i}{\partial x_j}\right|_A \tag{26}$$

where A denotes the anti-symmetric part. Using $\mathfrak{D}/\mathfrak{D}t$ to denote the Jaumann derivative, this determines (see Prager [9] p. 155)

$$\frac{\mathfrak{D}\overset{c}{\sigma}_{ij}}{\mathfrak{D}t} - \dot{\overset{c}{\sigma}}_{ij} = -\sigma_{ik}\left.\frac{\partial v_j}{\partial x_k}\right|_A - \sigma_{kj}\left.\frac{\partial v_j}{\partial x_k}\right|_A \tag{27}$$

Combination of (25), (27) and (6) gives

$$\dot{s}_{ij} = \left(\frac{\mathfrak{D}\overset{c}{\sigma}_{ij}}{\mathfrak{D}t} + \overset{c}{\sigma}_{ij}\frac{\partial v_k}{\partial x_k}\right) - (\sigma_{jk}D_{jk} + \sigma_{kj}D_{ik}) + \sigma_{ik}\frac{\partial v_j}{\partial x_k} \tag{28}$$

Equation (13) defines $\tau^{\alpha\beta}$, the Kirchhoff stress, in terms of convected coordinates. Being a tensor density, definition for other coordinates is obtained by use of (17) suitably modified to incorporate the density term J. We have seen that experiments from which the laws

of plasticity were defined involved "true stress" associated with Cartesian coordinates which, for plastic analyses, indicates the appropriateness of a Jaumann time-derivative associated with rotation of rectangular axes. Thus we need to utilize $\overset{c}{\tau}_{ij} = J \overset{c}{\sigma}_{ij}$ to incorporate the Jaumann derivative, and the first term in parenthesis in (28) can be written $\mathfrak{D}\overset{c}{\tau}_{ij}/\mathfrak{D}t$, since the derivative of the scalar density J is unaffected by rotation of axes, and at time t the instantaneous value of $J = 1$. Hill has stated ([illegible] p. 222) that the rate potential (19) implies a rate potential for the Jaumann derivative of $\overset{c}{\tau}_{ij}$, which provides computational advantages associated with the use of the Jaumann derivative of this stress variable.

Thus (28) takes the form:

$$\dot{s}_{ij} = \frac{\mathfrak{D}\overset{c}{\tau}_{ij}}{\mathfrak{D}t} - (\sigma_{ik} D_{jk} + \sigma_{kj} D_{ik}) + \sigma_{ik} \frac{\partial v_j}{\partial x_k} \tag{29}$$

Now combining (20) and (29)

$$\frac{\mathfrak{D}\overset{c}{\tau}_{ij}}{\mathfrak{D}t} = \dot{\tau}_{ij} + \sigma_{ik} D_{jk} + \sigma_{kj} D_{ik} \tag{30}$$

In view of (19) written for initial Cartesian coordinates, with $F(D_{ij})$ homogeneous of second order in the strain rates, multiplication of (30) by D_{ij} and using Euler's theorem gives

$$\frac{\mathfrak{D}\overset{c}{\tau}_{ij}}{\mathfrak{D}t} D_{ij} = 2F + \sigma_{ij} D_{jk} D_{ij} + \sigma_{kj} D_{ik} D_{ij} = 2F + 2\sigma_{ik} D_{ij} D_{jk} \tag{31}$$

which analogously to (21) for $\dot{s}^{\alpha\beta}$ provides a trial rate-potential function

$$G = F + \sigma_{ik} D_{ij} D_{jk} \tag{32}$$

assuming Euler's theorem. In a manner similar to that presented for $\dot{s}^{\alpha\beta}$, a third rate-potential function is thus established:

$$\frac{\mathfrak{D}\overset{c}{\tau}_{ij}}{\mathfrak{D}t} = \frac{\partial G}{\partial D_{ij}} \tag{33}$$

The classical elastic-plastic isotropically work hardening law (Prandtl-Reuss), commonly giving strain-rate as a linear function of stress-rate, can be inverted to give stress-rate and takes the form ([5] p. 606)

$$\frac{\mathfrak{D}\overset{c}{\sigma}_{ij}}{\mathfrak{D}t} = \frac{E}{1+\nu}\left[\delta_{ik}\delta_{jl} + \frac{\nu}{1-2\nu}\delta_{ij}\delta_{kl} - \frac{3\overset{c}{\sigma}'_{ij}\,\overset{c}{\sigma}'_{kl}\left(\frac{E}{1+\nu}\right)}{2\bar{\sigma}^2\left(\frac{2}{3}h + \frac{E}{1+\nu}\right)}\right] D_{kl} \tag{34}$$

where E and ν are Young's modulus and Poisson's ratio, σ' denotes stress deviator, $\bar{\sigma}$ is the current tensile yield stress and h the current gradient of the true-stress logarithmic plastic strain curve in a tension test. The Jaumann derivative is used for stress-rate as mentioned above in order to eliminate rotation effects, and the last term in the brackets is dropped when the increment of deformation is elastic. We have shown that there is a rate-potential function G for $\overset{c}{\tau}_{ij}$, but not one for $\overset{c}{\sigma}_{ij}$. This means [5] that a non-symmetric finite-element stiffness matrix would be deduced using (34). Replacing $\overset{c}{\sigma}_{ij}$ by $\overset{c}{\tau}_{ij}$ in the stress-rate term would yield a symmetric stiffness matrix which would simplify the numerical analysis. Moreover such a change is appropriate in terms of its representation of the physical laws. The J term in

$$\overset{c}{\tau}_{ij} = J\,\overset{c}{\sigma}_{ij} \tag{35}$$

arises in more accurate representation of the laws of elasticity than Hooke's law used in classical elasticity. It is associated with geometrical non-linearity which expresses the non-linear influence of finite strain. In effect it expresses the fact that energy density per unit initial volume yields a simpler energy conservation statement, for per unit current volume implies change of energy density simply because the amount of material containing it changes. For example, the term (ρ/ρ_o) appearing in equation (76) of [10] is equivalent to replacing $\underset{\sim}{\sigma}$ by $J\underset{\sim}{\sigma}$, and it was pointed out in that paper (p. 935) that such a term provides a good approximation to non-linear elasticity of metals with input of only the two classical elastic constants. A similar modification of the classical laws of plasticity was suggested in [11] where it was found compelling to express the yield stress in terms of $J\underset{\sim}{\sigma}$ (equs. (33) and (34) of [11]). Again this was based on the requirements of geometrical non-linearity, which of course are independent of specific material characteristics.

Introduction of (2) into the variational principle (3) expresses it in the form

$$\int_{\mathring{V}} \dot{s}_{ij}\ \delta\left(\frac{\partial v_i}{\partial x_i}\right) d\mathring{V} - \delta\left(\int_{\mathring{S}_F} \dot{F}_j v_j d\mathring{S} + \int_{\mathring{V}} \dot{g}_j v_j d\mathring{V}\right) = 0 \tag{36}$$

which, since the reference state is the current state ($\underset{\sim}{X} = \underset{\sim}{x}$) can be written

$$\int_V \dot{s}_{ij}\ \delta\left(\frac{\partial v_i}{\partial x_i}\right) dV - \delta\left(\int_{S_F} \dot{F}_j v_j dS + \int_V \dot{g}_j v_j dV\right) = 0 \tag{37}$$

Substitution of (29) and applying algebraic manipulation based on

symmetries then yields the variational principle in the form (see e.g. (5) of [5])

$$\int_V \left[\frac{\mathfrak{D}\overset{c}{\tau}_{ij}}{\mathfrak{D}t}\delta(D_{ij}) - \tfrac{1}{2}\sigma_{ij}\delta(2D_{ik}D_{kj} - v_{k,i}v_{k,j})\right]dV$$

$$-\delta\left(\int_{S_F}\dot{F}_j v_j dS + \int_V \dot{g}_j v_j dV\right) = 0 \tag{38}$$

where the subscript $,i$ denotes the operation $\partial/\partial x_i$. Now utilizing (33), the principle takes the form (see equ. (15) of [5]).

$$\delta\left[\int_V G(D)\,dV - \tfrac{1}{2}\int_V \sigma_{ij}(2D_{ik}D_{kj} - v_{k,i}v_{k,j})dV\right.$$

$$\left.- \int_{S_F}\dot{F}_j v_j dS - \int_V \dot{g}_j v_j dV\right] = 0 \tag{39}$$

By (33), and using Euler's theorem

$$\frac{\mathfrak{D}\overset{c}{\tau}_{ij}}{\mathfrak{D}t} = \frac{\partial G}{\partial D_{ij}} = \frac{\partial^2 G}{\partial D_{ij}\partial D_{kl}} D_{kl} = \mathfrak{L}_{ijkl} D_{kl} \tag{40}$$

since $\partial G/\partial D_{ij}$ is homogeneous of first degree in $\underset{\sim}{D}$, hence $\mathfrak{L}_{ijkl}$ is symmetric in $ij \leftrightarrow kl$ as well as $i \leftrightarrow j$ and $k \leftrightarrow l$. Since G is homogeneous of second degree

$$G(\underset{\sim}{D}) = \tfrac{1}{2}\frac{\mathfrak{D}\overset{c}{\tau}_{ij}}{\mathfrak{D}t} D_{ij} \tag{41}$$

The symmetry implicit in the variational principles (38) and (39) with (40) and (41) imply symmetric stiffness matrices which carry through to the finite-element formulation [5].

3. Discussion

As discussed by Rice [4] and McMeeking and Rice [5], the development reviewed in these notes brings out the importance of convection

effects and appropriate stress-rate definitions in formulating elastic-plastic theory. Thus the precise analysis of continuum theory must be applied to obtain reliable results. Common small strain assumptions are not valid even for incremental theory based on small strain increments. This situation is implicit in comparison of the first and second terms in the first volume integral in the variational principle (38). In plastic flow the coefficient of $\delta \underset{\sim}{D}$ in the first term is $\sim h\underset{\sim}{D}$, where h is the gradient of the tensile stress-plastic strain curve, whereas the second term is $\sim \underset{\sim}{\sigma}\underset{\sim}{D}\delta\underset{\sim}{D}$. The second term and the difference between the Jaumann derivative and other simpler time derivatives can only be neglected if $h \gg \underset{\sim}{\sigma}$. The relative error in neglecting such terms is effectively independent of the magnitude of the strain increment adopted, so that small steps do not permit simplification in this regard. For many metals $h \sim \sigma$.

It is interesting to note that the complications which arise in elastic-plastic analysis result from the elastic component of strain, and not the plastic. For stress and strain deviators, elastic-ideally plastic deformation with a Mises yield condition, $J_2 = \sigma'_{ij}\sigma'_{ij}/2 = k^2$, satisfies

$$D'_{ij} = \dot{\sigma}'_{ij}/2G + \dot{\lambda}\sigma'_{ij} \tag{42}$$

where $\dot{\lambda}$ is a parameter. Multiplication of (42) by σ'_{ij} gives

$$\dot{\lambda} = \sigma'_{ij}D'_{ij}/2J_2 \tag{43}$$

For isotropic work hardening with a Mises yield condition, (42) takes the form

$$D'_{ij} = \dot{\sigma}'_{ij}/2G + f(J_2)\dot{J}_2\sigma'_{ij} \tag{44}$$

The first terms on the right hand sides of (42) and (44) are the elastic strain rate components. Thus rigid-ideally plastic theory ((42) with the $\dot{\sigma}$ term deleted) gives a relation between stress and velocity gradient with no complications due to a stress rate term, which greatly simplifies the analysis, apart from the difficulty of determining the rigid regions. Similarly for work hardening rigid-plastic analysis ((44) with the $\dot{\sigma}$ term deleted), only the rate of change of a stress invariant occurs, which is simpler to include than a tensor rate. For elastic-plastic theory it is the elastic term which introduces stress-rate and the consequent complications.

Many technologically important metal-forming problems are steady state processes in which $\partial\sigma_{ij}/\partial t|_{\underset{\sim}{x}} = 0$. In planning to use an analysis of the type considered here to evaluate such situations, it is fortunate that the stress-rate terms appearing in the variational principle, $\dot{s}_1$, or $\mathcal{D}\overset{c}{\tau}_{ij}/\mathcal{D}t$, do not approach zero in the steady case, hence singular computational conditions need not be anticipated. This is not the case in some simpler and inadequate approaches to this problem in which sufficient care was not devoted to the appropriate choice of stress-rate definition.

On the basis of the theory described, a finite-element program has been written to evaluate stress and deformation histories in an extrusion problem. The case of a billet being pushed through a die until a steady-state configuration was reached has been completed.

The stress field exhibits features which are consistent with the known development of extrusion defects, such as the appearance of surface cracks.

4. Acknowledgement

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6. Appendix

For simplicity, relations are developed for Cartesian axes in the reference state which is instantaneously coincident with the current state. The general theory for convected coordinates carries through in a similar manner but can be technically much more involved.

Equations (16) and (24) yield the relation,

$$\underset{\sim}{\tau}\,\underset{\sim}{F}^{T} = \underset{\sim}{s} \tag{A1}$$

Material differentiation of this relation gives

$$\dot{\underset{\sim}{\tau}}\,\underset{\sim}{F}^{T} + \tau\,\dot{F}^{T} = \dot{s}$$

and for the special reference configuration

$$\underset{\sim}{F} = \underset{\sim}{1}\ , \quad \dot{F} = \frac{\partial v_i}{\partial X_j} = \frac{\partial v_i}{\partial x_j}\ , \quad \text{hence} \quad \dot{s}_{ij} = \dot{\tau}_{ij} + \tau_{ik}\frac{\partial v_j}{\partial x_k} \tag{A2}$$

Figure 1. Convected coordinates, $\xi^{\alpha} = \xi^{\alpha'}$.

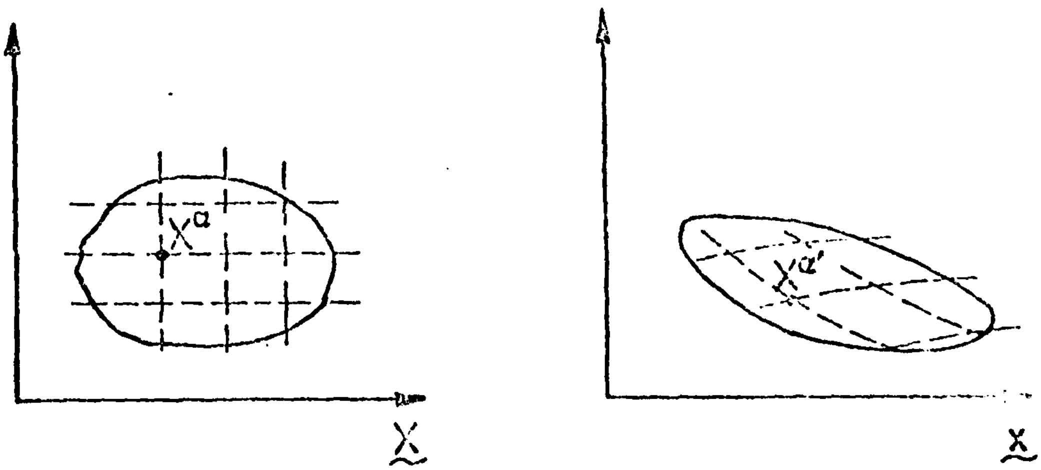

Figure 2. Cartesian reference coordinates, X^{α} .

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